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# $AdS/CFT$ correspondence and quotient space geometry

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Abstract: We consider a version of the $AdS_{d+1}/CFT_d$ correspondence, in which the bulk space is taken to be the quotient manifold $AdS_{d+1}/\Gamma$ with a fairly generic discrete group $\Gamma$ acting isometrically on $AdS_{d+1}$. We address some geometrical issues concerning the holographic principle and the UV/IR relations. It is shown that certain singular structures on the quotient boundary $\mathbf{S}^d/\Gamma$ can affect the underlying physical spectrum. In particular, the conformal dimension of the most relevant operators in the boundary CFT can increase as $\Gamma$ becomes "large". This phenomenon also has a natural explanation in terms of the bulk supergravity theory. The scalar two-point function is computed using this quotient version of the $AdS/CFT$ correspondence, which agrees with the expected result derived from conformal invariance of the boundary theory.

Keywords: M-Theory, String Duality, Brane Dynamics in Gauge Theories.

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## Contents



## 1. Introduction

The $AdS/CFT$ correspondence proposed in [1, 2, 3] gives a surprisingly powerful method to investigate strong coupling physics. In its simplest form, this proposal relates tree-level supergravity on $(d+1)$-dimensional anti-de Sitter space to a conformal field theory on the round sphere $\mathbf{S}^d$, which is the conformal boundary of $AdS_{d+1}$. One may consider CFT on a more complicated manifold $M$ as well, by replacing the bulk spacetime $AdS_{d+1}$ with a negatively curved Einstein space $X$ of boundary $M$. In fact, using topologically non-trivial $X$ can provide additional tests for the bulk/boundary correspondence [4, 5, 6].

There are several ways of constructing a non-trivial Einstein manifold with negative cosmological constant. One way is to place black holes in $AdS$ space. This amounts to a study of the boundary theory at finite temperature [4]. While this construction can exhibit interesting phase structures in the large $N$ limit (using a mechanism discovered by Hawking and Page [7] many years ago), finding black hole solutions in the $AdS$ background is however by no means an easy task; at present only a few such solutions are explicitly known.

Alternatively, we can pick a discrete subgroup $\Gamma$ of $\mathrm{SO}(d,2)$ that acts isometrically on $AdS_{d+1}$, and take the quotient manifold $X = AdS_{d+1}/\Gamma$ to be the bulk spacetime. The resulting space will constitute a $(d+1)$-dimensional negatively curved

Einstein manifold, on which the classical supergravity action can be defined. One thus expects that gravity in $X$ still corresponds to some conformal field theory on the quotient boundary $M = \mathbf{S}^d/\Gamma$. For a check on this quotient version of the $AdS/CFT$ correspondence, recall that the Hawking-Page reference space (denoted by $X_1$ in [3, 4]) takes the form $AdS_{d+1}/\mathbb{Z}$, which is associated to a boundary CFT in low-temperature phase. Another useful check concerns a quotient construction of the BTZ black hole [8, 9] in $AdS_3$, where $\Gamma$ is generated by a single loxodromic element; supergravity in this geometry is again related to a $2D$ CFT (see e.g. [10] for such a study and the references therein). Note that these examples made a common choice of $\Gamma$ in elementary discrete groups.

In the present work we wish to consider some general features of the quotient $AdS/CFT$ correspondence. We will address a couple of geometrical issues that are closely related to the holographic principle [11] and the UV/IR relations [12, 13] between bulk- and boundary- theories. As we will see in detail, certain singular structures on the quotient boundary $\mathbf{S}^d/\Gamma$ may affect the underlying physical spectrum. This effect could not be seen by choosing $\Gamma$ in elementary discrete groups; mathematically such a choice is too special to describe generic structures of quotient manifolds. Thus, in this paper we will take $\Gamma$ to be non-elementary, namely it contains no abelian subgroups of finite index. With this choice the conformal dimension of the most relevant operators in the boundary CFT will depend nontrivially on $\Gamma$, provided $\Gamma$ is "large" enough. We will give some mathematical explanations of such a dependence, both on the supergravity side and from the boundary point of view. According to this dependence we could have, for $\Gamma$ extremely large, a boundary theory that does not contain any relevant operators, — corresponding to a bulk supergravity theory without tachyon modes.

The existence of singular structures on the boundary and its relations to the bulk geometry have been extensively studied by mathematicians. Some of the mathematical results will be reviewed in this paper. Section 2 provides a brief description of the quotient space geometry. In section 3, we discuss several implications of the holographic principle as well as certain geometrical subtleties arising from constructing the quotient boundary. Section 4 then deals with scalar conformal fields on the quotient boundary, where we derive a constraint on the spectrum of conformal dimensions. In section 5 we compute the scalar two-point function using the quotient $AdS/CFT$ correspondence. Finally, after presenting our conclusions in section 6, we give an appendix to clarify some points in the text.

## 2. Taking the quotient

We will acturally work in Euclidean language and take the covering space of our bulk to be the $(d+1)$-dimensional hyperbolic space $\mathbf{H}^{d+1} \cong \mathrm{SO}(d+1,1)/\mathrm{SO}(d+1)$. This is a complete, simply connected Riemannian manifold with negative constant curvature and having $\mathrm{SO}(d+1,1)$ as its isometry group. A particularly useful description of $\mathbf{H}^{d+1}$ is the open half-space model, which has the standard hyperbolic metric $ds^2 = |dx|^2/x_0^2$ and the boundary sphere

$$\mathbf{S}^d \;\simeq\; \mathbf{R}^d \cup \{\infty\} \;\equiv\; \hat{\mathbf{R}}^d \tag{2.1}$$

at infinity, here the $d$-dimensional plane $\mathbf{R}^d$ is specified by the equation $x_0 = 0$.

Geometrically, there is a simple realization of isometric transformations in this half-space model [14]. Suppose that $\mathcal{S}$ is a $d$-dimensional Euclidean hemisphere in the half-space whose equatorial plane coincides with the boundary plane $\mathbf{R}^d$ in (2.1). The inversion in $\mathcal{S}$, which will be denoted by $\mathcal{I}_{\mathcal{S}}$, is an orientation-reversing isometry of $\mathbf{H}^{d+1}$. In the limiting case when $\mathcal{S}$ approaches a $d$-dimensional Euclidean half-plane in $\mathbf{H}^{d+1}$ perpendicular to the boundary plane, the inversion $\mathcal{I}_{\mathcal{S}}$ becomes an ordinary reflection, which again is an orientation-reversing isometry. To obtain an orientation-preserving isometry, one simply takes product of an even number of such $\mathcal{I}_{\mathcal{S}}$'s. It turns out that each isometry of the hyperbolic space can be realized in this way, namely for any $g \in \mathrm{SO}(d+1,1)$, we can find $2n$ hemispheres $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_{2n}$ as described above (possibly including those in the Euclidean half-plane limit), so that

$$g \;=\; \mathcal{I}_{\mathcal{S}_1}\mathcal{I}_{\mathcal{S}_2}\ldots\mathcal{I}_{\mathcal{S}_{2n}}\,. \tag{2.2}$$

Note that each inversion $\mathcal{I}_{\mathcal{S}} : \mathbf{H}^{d+1} \to \mathbf{H}^{d+1}$ on the hyperbolic space induces a transformation $\mathcal{I}_{\partial\bar{\mathcal{S}}} : \mathbf{S}^d \to \mathbf{S}^d$ on the boundary (2.1), defined by the inversion in $\partial\bar{\mathcal{S}}$, where $\bar{\mathcal{S}}$ is the closure of $\mathcal{S}$ whose boundary $\partial\bar{\mathcal{S}}$ is a $(d-1)$-dimensional Euclidean sphere (or Euclidean plane in the limiting case) living in (2.1). Thus group elements of the form (2.2) also act naturally on the boundary sphere, giving the so-called Möbius transformations [14].

Now we choose a discrete subgroup $\Gamma$ of $\mathrm{SO}(d+1,1)$. Discreteness means that each $\gamma \in \Gamma$ has a neighborhood $U_\gamma \subset \mathrm{SO}(d+1,1)$ such that $U_\gamma \cap \Gamma = \{\gamma\}$. So every sequence of infinitely many distinct elements $\{\gamma_n\}$ in $\Gamma$ cannot converge to any point of $\mathrm{SO}(d+1,1)$ and, in particular, we have $\lim_{n\to\infty} ||\gamma_n|| = \infty$. Evidently, any element of $\Gamma$ has a product represention (2.2) in terms of inversions. This specifies how $\Gamma$ acts on $\mathbf{H}^{d+1}$ in the half-space description. The $(d+1)$-dimensional bulk spacetime $X$ is then constructed by

$$X = \frac{\mathbf{H}^{d+1}}{\Gamma}. \tag{2.3}$$

If $\Gamma$ does not contain elements of finite order, then it acts freely on $\mathbf{H}^{d+1}$, so that $X$ is a smooth hyperbolic manifold without orbifold singularities.

Since $\Gamma$ acts isometrically on $\mathbf{H}^{d+1}$, the local geometry of $X$ looks the same as the original hyperbolic geometry. In particular $X$ inherits the standard hyperbolic metric from $\mathbf{H}^{d+1}$. Of course, taking the quotient (2.3) will generally break the isometry group SO$(d+1,1)$. The residual spacetime symmetry on $X$ will be determind by:[1]

$$G = \{g \in \mathrm{SO}(d+1,1) \mid g \cdot \gamma \cdot g^{-1} \in \Gamma,\ \forall \gamma \in \Gamma\}. \tag{2.4}$$

In the quotient version of the $AdS/CFT$ correspondence, this isometry group should be identified with the unbroken conformal symmetry of the boundary theory. An independent verification of this fact will be given at the end of section 5.

From the boundary point of view, it is natural to regard $\Gamma$ as a discrete group of certain Möbius transformations acting on the boundary sphere $\mathbf{S}^d$. Thus the "quotient boundary" is constructed by

$$M = \mathbf{S}^d/\Gamma\,. \tag{2.5}$$

In contrast to the previous case, $M$ in general does not inherit any metric from $\mathbf{S}^d$, as generic Möbius transformations cannot preserve such a metric. Nevertheless, this quotient space *does* inherit a conformal structure from $\mathbf{S}^d$.

## 3. Holography and singular structures

The usual $AdS/CFT$ correspondence associates each (on-shell) bulk degree of freedom on $\mathbf{H}^{d+1}$ to a certain boundary degree of freedom on $\mathbf{S}^d$. When taking the quotient by $\Gamma$, bulk fields in $X = \mathbf{H}^{d+1}/\Gamma$ can be obtained by $\Gamma$-invariant projection of those in the original $AdS$ space, and the resulting theory may be considered as a truncated version of the ordinary $AdS$ supergravity, in which only $\Gamma$-invariant degrees of freedom are allowed. The projection also forces boundary values of a bulk field to be invariant under $\Gamma$, which, in turn, give rise to a $\Gamma$-invariant dual image in the boundary CFT defined on the original sphere $\mathbf{S}^d$, via the usual $AdS/CFT$ duality. The collection of all such dual images then forms the field content of a truncated boundary theory. A naive generalization of the $AdS/CFT$ correspondence will thus provide a connection between the truncated bulk- and boundary- theories. Although this could not be checked directly by taking the near-horizon limit (a generic $\Gamma$ does not preserve the metrics generated by D brane sources, so taking the quotient in general cannot commute with taking the near-horizon limit), explicit computations of correlators support such a generalization; see appendix A for a detailed discussion.

At this point, one might expect that supergravity on the quotient bulk $X$ should correspond to a conformal field theory living in the $d$-dimensional space $M$, defined by (2.5), since the latter theory (if exists) seems to provide the natural $\Gamma$-invariant

[1] A quick derivation of this group is as follow. Each point $x$ of $X$ can be considered as an orbit $[y] \equiv \Gamma \cdot y$ in $\mathbf{H}^{d+1}$, starting at some $y$ in the hyperbolic space. The action of $g \in \mathrm{SO}(d+1,1)$ on $X$ is defined by $x \mapsto g \cdot x \equiv [g \cdot y]$, which is well-defined provided $[g \cdot y]$ is independent of the choice of representive $y$ in the orbit; thus, if $y' = \gamma \cdot y$ is another representive, there should exist a $\gamma' \in \Gamma$ such that $g \cdot y' = \gamma' \cdot (g \cdot y)$. The collection of all such $g$ is precisely the isometry group of $X$.

truncation of the original boundary theory on $\mathbf{S}^d$. The geometry of the quotient boundary space (2.5) is quite clear in examples with elementary discrete groups [3, 4, 10]. For generic $\Gamma$, however, there may exist some subtleties that at first sight will raise a puzzle, — ignoring such subtleties and simply supposing $M$ to be the space on which the holographic boundary theory lives would violate the UV/IR relations [3, 12, 13] between boundary- and bulk- theories and thus conflict with the holographic principle proposed in [11]. We will see how this can happen in the following extreme case.

Generally speaking, taking the quotient of $\mathbf{H}^{d+1}$ by a discrete group $\Gamma$ will cause reduction of the spacetime volume $V \to V/|\Gamma|$. For finite groups this reduction is invisible because of $V = \infty$ in the hyperbolic metric. But when $\Gamma$ is chosen to be large enough, we can actually make the quotient space volume $\mathrm{Vol}(X)$ finite. In this case $X$ may be viewed as a regulated version of the original $AdS$-space $\mathbf{H}^{d+1}$, with a certain $\Gamma$-invariant infrared cutoff imposed. In other words, one regulates the bulk theory by putting it into some finite-volume box — a fundamental domain for $\Gamma$, and imposing the "periodic" conditions compatible with the action of $\Gamma$ on each side of that box. The infrared cutoff may be thought of as the mean length scale of the fundamental domain, thus having a $\Gamma$-invariant meaning. On the other hand, we may start from a classical conformal field theory defined on the original $AdS$-boundary $\mathbf{S}^d$. The quantum version of this theory can be regulated by an ultraviolet cutoff so that it lives on a discrete subspace of $\mathbf{S}^d$. There may exist many different lattice models of the boundary sphere, but according to the UV/IR relations [3, 12, 13], only one such discrete subspace $M'$ can correspond to the given infrared regulating $\mathbf{H}^{d+1} \rightsquigarrow X$ of the bulk, as long as $\mathrm{Vol}(X) < \infty$.[2] Since this $M'$ arises from an ultraviolet cutoff of $\mathbf{S}^d$ while $M = \mathbf{S}^d/\Gamma$ does not, in general we should have $M' \neq M$ and, since the cutoff has a $\Gamma$-invariant meaning, $M'$ is likely to be a subset of $M$. Thus, supergravity on the quotient bulk space $X$ should generally correspond to a "boundary" theory on $M'$ other than the quotient boundary (2.5).

There is an interesting consequence of the above discussion. If $\mathrm{Vol}(X) < \infty$, the ultraviolet cutoff completely breaks the conformal algebra on $\mathbf{S}^d$, so holography predicts that the residual symmetry (2.4) will at most constitute a discrete group. This is in accordance with the mathematical speculation that generic hyperbolic manifolds with finite volumes admit no continuous isometries. Continuous isometries can arise only if the volume of $X$ is infinite.

So far we have only considered an extreme case that can occur in the $AdS/CFT$ correspondence: When $\mathrm{Vol}(X) < \infty$, one naively expects that conformal fields should be defined on the quotient boundary $M = \mathbf{S}^d/\Gamma$, but turning to the quantum theory

[2] If $\mathrm{Vol}(X) = \infty$, $X$ does not really regulate $\mathbf{H}^{d+1}$ through an infrared cutoff, so $M'$ needs not to be discrete.



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a rather intriguing, apparantly discrete space $M' \subset M$ can possibly emerge as the regulated spacetime. It would be of interest to have a geometrical understanding of why the boundary of $X$ can be discretized as one deforms $\Gamma$ from generic position into this extreme case.

The crucial point is that the action of a discrete $\Gamma \subset \mathrm{SO}(d+1,1)$ on $\mathbf{H}^{d+1}$ may behave very differently from that on the boundary sphere $\mathbf{S}^d$. One can show that $\Gamma$ always acts discontinuously[3] on $\mathbf{H}^{d+1}$. In fact, if this is not the case, then we can find certain $x \in \mathbf{H}^{d+1}$ and a sequence of infinitely many distinct elements $\{\gamma_n\}$ in $\Gamma$, such that $\gamma_n(x)$ coverges to some $y \in \mathbf{H}^{d+1}$. Since $\mathbf{H}^{d+1}$ can be regarded as the coset space $\mathrm{SO}(d+1,1)/\mathrm{SO}(d+1)$, the points $x$ and $y$ in the hyperbolic space may be represented by some group elements $g_x \sim g_x \cdot h_x$ and $g_y \sim g_y \cdot h_y$ in $\mathrm{SO}(d+1,1)$, respectively, up to arbitrary $h_x, h_y \in \mathrm{SO}(d+1)$. Thus, the limit $\gamma_n(x) \to y$ can be alternatively described by $\gamma_n \cdot g_x = g_{\gamma_n(x)} \cdot h_n$ with $g_{\gamma_n(x)} \to g_y$ and $h_n \in \mathrm{SO}(d+1)$. This clearly gives $\gamma_n = g_{\gamma_n(x)} \cdot h_n \cdot g_x^{-1}$. Now because $g_{\gamma_n(x)}$ has a finite limit $g_y$ as $n \to \infty$, and because the $h_n$'s belong to a compact group $\mathrm{SO}(d+1)$, we see that $||\gamma_n||$ is bounded from above. But this is impossible due to discreteness of $\Gamma$. So as far as $\Gamma$ is discrete, its action on $\mathbf{H}^{d+1}$ is necessarily discontinuous[4]. Accordingly the quotient space $X = \mathbf{H}^{d+1}/\Gamma$ with its natural quotient topology is always a Hausdorff space.

On the contrary, a discrete group $\Gamma$ in general does not act discontinuously on the boundary of $\mathbf{H}^{d+1}$ and there may exist accumulation (or cluster) points in each $\Gamma$-orbit, namely, given any $x$ on the boundary sphere $\mathbf{S}^d$, one may find an infinite sequence $\{\gamma_n\}$ in $\Gamma$ such that $\lim_{n\to\infty} \gamma_n(x) \in \mathbf{S}^d$. These points may render the quotient space $\mathbf{S}^d/\Gamma$ possible to be non-Hausdorff. The set of all such accumulation points is called the limit set $\Lambda_\Gamma$ of $\Gamma$. This set is a closed, $\Gamma$-invariant subset of $\mathbf{S}^d$; actually it is the smallest non-empty subset having such properties. Thus $\Lambda_\Gamma$ is contained in the closure of an orbit $\Gamma \cdot x$, with $x$ either on the boundary sphere or in the interior of the hyperbolic space. Note that if $\Gamma' \subset \Gamma$, then $\Lambda_{\Gamma'} \subset \Lambda_\Gamma$. Hence a "large" $\Gamma$ actually means that $\Lambda_\Gamma$ has a relatively large Hausdorff dimension.

Let us give an example to illustrate how the limit set of $\Gamma$ can have positive Hausdorff dimension. This example comes from a slight modification of the material exposed in [15]. As we have mentioned before, the action of $\Gamma$ on the hyperbolic space is generated by inversions with respect to some $d$-dimensional Euclidean hemispheres (or half-planes) in $\mathbf{H}^{d+1}$. We restrict ourselves to a simple case in which only finitely many such hemispheres can occur. Thus, for $i = 1, 2, \ldots, K+1$, let $\mathcal{S}_i$ be a $d$-dimensional Euclidean hemisphere with center $a_i$ and Euclidean radius $r_i$, and let

[3] We say that $\Gamma$ acts discontinuously on a space $Y$ provided for every compact subset $K$ of $Y$, the condition $\gamma(K) \cap K = \emptyset$ holds except for finitely many $\gamma \in \Gamma$. Thus a finite group acts on $Y$ discontinuously. If $\Gamma$ is infinite and acts discontinuously on $Y$, then for all $y \in Y$ as well as $\gamma_1, \gamma_2, \ldots \in \Gamma$, the sequence $\{\gamma_n(y)\}$ cannot converge to any point of $Y$. See section 5.3 of [14] for a more detailed discussion.

[4] See [14] for a somewhat different proof of this result in the case $d = 2$.

$\mathcal{B}_i$ be the $(d+1)$-dimensional Euclidean half-ball bounded by $\mathcal{S}_i$, whose closure is denoted by $\bar{\mathcal{B}}_i$. Suppose that these half-balls are mutually disjoint, namely for each pair $(i,j)$, $|a_i - a_j| > r_i + r_j$. Then the discrete group $\Gamma$ which we will consider consists of all elements of the form

$$\gamma_{i_1 i_2 \ldots i_{2n}} = \mathcal{I}_{i_1}\mathcal{I}_{i_2}\ldots\mathcal{I}_{i_{2n-1}}\mathcal{I}_{i_{2n}}\,, \qquad (i_1, i_2, \ldots, i_{2n}) \in \Sigma(2n)\,, \tag{3.1}$$

where $\mathcal{I}_i$ is the inversion in $\mathcal{S}_i$, $\Sigma(m)$ denotes the set of all sequences $i_1, \ldots, i_m$ such that $1 \le i_1, i_2, \ldots, i_m \le K+1$ and $i_1 \ne i_2, i_2 \ne i_3, \ldots, i_{m-1} \ne i_m$ (that is, adjacent indices have different values). Since all the half-balls do not touch each other, we have $\mathcal{I}_i(\bar{\mathcal{B}}_j) \subset \bar{\mathcal{B}}_i$ for $i \ne j$, and therefore

$$\bar{\mathcal{B}}_{i_1 i_2 \ldots i_{2n+1}} \equiv \gamma_{i_1 i_2 \ldots i_{2n}}(\bar{\mathcal{B}}_{i_{2n+1}}) \subset \bar{\mathcal{B}}_{i_1}\,, \qquad \forall (i_1, i_2 \ldots i_{2n+1}) \in \Sigma(2n+1)\,. \tag{3.2}$$

Moreover, it is not difficult to see that (3.2) itself is a closed half-ball in $\mathbf{H}^{d+1} \cup \mathbf{S}^d$, with the center $a_{i_1 i_2 \ldots i_{2n+1}}$ and radius $r_{i_1 i_2 \ldots i_{2n+1}}$ determined iteratively by

$$a_{i_1 i_2 \ldots i_m} = a_{i_1} - \frac{r_{i_1}^2\,(a_{i_1} - a_{i_2 \ldots i_m})}{|a_{i_1} - a_{i_2 \ldots i_m}|^2 - r_{i_2 \ldots i_m}^2}\,, \qquad r_{i_1 i_2 \ldots i_m} = \frac{r_{i_1}^2\, r_{i_2 \ldots i_m}}{|a_{i_1} - a_{i_2 \ldots i_m}|^2 - r_{i_2 \ldots i_m}^2}\,, \tag{3.3}$$

where $m \ge 2$ and $(i_1, i_2, \ldots, i_m) \in \Sigma(m)$. It follows from the (geometrically obvious) inequalities $|a_{i_1} - a_{i_2 \ldots i_m}|^2 - r_{i_2 \ldots i_m}^2 \ge (|a_{i_1} - a_{i_2 \ldots i_m}| - r_{i_2 \ldots i_m})^2 \ge (|a_{i_1} - a_{i_2}| - r_{i_2})^2$ that the radius of $\bar{\mathcal{B}}_{i_1 i_2 \ldots i_{2n+1}}$ will shrink to zero when $n$ tends to infinity. Thus, the following closed set

$$E = \bigcap_{n=1}^{\infty} \bigcup_{(i_1 i_2 \ldots i_{2n+1}) \in \Sigma(2n+1)} \bar{\mathcal{B}}_{i_1 i_2 \ldots i_{2n+1}} \tag{3.4}$$

is a subset of the boundary sphere $\mathbf{S}^d$ and, by construction, it is the smallest non-empty closed subset of $\mathbf{S}^d$ invariant under the action of $\Gamma$. So $E$ is precisely the limit set $\Lambda_\Gamma$. Since (3.4) constitutes a kind of the "general Cantor sets" [16], its Hausdorff dimension $d_H(E)$ obeys [15]

$$d_H(E) \ \ge C_K \log K \tag{3.5}$$

where $C_K$ is some positive constant. Thus, for $K \ge 2$, the limit set of $\Gamma$ indeed has a positive Hausdorff dimension. (In the case $K = 1$ when $\Gamma$ becomes an elementary group, we have $d_H(E) = 0$ and $E$ consists only of two points.)

Returning to generic case, we see that $M = \mathbf{S}^d/\Gamma$ is not a Hausdorff space. It is not quite clear how to quantize classical conformal fields on a non-Hausdorff space. So when we try to associate a quantized "boundary theory" to (semi-) classical gravity on the bulk spacetime $X$, we have to discard the singular points on $M$ that cause the quotient boundary to be non-Hausdorff, and consider the maximal subset $M'$ of $M$ such that $M'$ is topologically a Haustorff space. This naturally leads us to the notation of *domain of discontinuity*, $D_\Gamma$, defined as the $\Gamma$-invariant open set $D_\Gamma = \mathbf{S}^d - \Lambda_\Gamma$. The group $\Gamma$ now acts properly disontinuously on $D_\Gamma$ and we propose that the "regulated boundary" $M'$ is to be constructed as

$$M' = D_\Gamma/\Gamma, \tag{3.6}$$

plus possible cusps of $\Gamma$ living outside $D_\Gamma$. We will call such $M'$ as the "Kleinian boundary" of $X$. Geometrically, the quotient (3.6) can be interpreted as the boundary of the Kleinian manifold $(\mathbf{H}^{d+1} \cup D_\Gamma)/\Gamma$ [17]. Any point on $\mathbf{S}^d$ living in the limit set will come from an interior point $x$ of $\mathbf{H}^{d+1}$ under a certain limit $\lim_{n\to\infty} \gamma_n(x)$, $\gamma_n \in \Gamma$. So when taking the quotient (2.3) by $\Gamma$, a point $y \in \Lambda_\Gamma \subset \mathbf{S}^d$ can hardly be thought of as a boundary point of the quotient space $X$, unless $y$ is a cusp.

Thus, if $\Gamma$ contains no parabolic elements or, more generally, if $\Gamma$ has a fundamental domain in $\mathbf{H}^{d+1}$ that does not touch the limit set (so that all possible cusps are in $D_\Gamma$), then the Kleinian boundary of $X$ is just the $d$-dimensional Hausdorff space (3.6). On the other hand, if $\Gamma$ has cusps not living in $D_\Gamma$, they should be regarded as additional points of the Kleinian boundary $M'$. These additional points may become important when $\Lambda_\Gamma$ is so large that all the connected components of $D_\Gamma$ collapse. In that case the Hausdorff dimension of the limit set reaches its maximal value $d$ and $M'$ is discretized or even becomes empty. Typically this can occur when $X$ has finite volume or is compact. We thus find a geometrical resolusion of the apparent puzzle mentioned at the begining of this section. The puzzle can be removed simply by requiring that the quotient version of the $AdS/CFT$ correspondence should relate supergravity in $X$ to a holographic theory defined on the Kleinian boundary $M'$, and not on the naive quotient space $M = \mathbf{S}^d/\Gamma$.

We now elucidate the above discussions by some examples. The most familiar example concerns a finitely generated Fuchsian group of the first kind [18]. This is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ acting on $\mathbf{H}^2$ as well as on its boundary $\hat{\mathbf{R}}$, whose limit set $\Lambda_\Gamma$ turns out to be the whole $\hat{\mathbf{R}}$ and hence the domain of discontinuity $D_\Gamma$ is empty. To get an intuitive picture of what the "Kleinian boundary" of the Riemann surface $X = \mathbf{H}^2/\Gamma$ should look like, let us consider the typical case where the canonical fundamental domain $\mathcal{F}$ for $\Gamma$ has a finite hyperbolic volume. Then according to [18], $\mathcal{F}$ has $4g + s + p$ (finitely many) vertices, $4g$ of which are used to create genus of $X$, $s$ of which correspond to elliptic fixed points, and $p$ of which sit on $\hat{\mathbf{R}}$, corresponding to cusps. Each elliptic fixed point is associated to a finite cyclic subgroup of $\Gamma$, hence giving rise to an orbifold singularity in the interior of the Riemann surface. If we pick local coordiates $z$ (with $|z| < 1$) at an elliptic fixed point, then a locally analytic function $f$ will behave as $f(z) \sim z^{1/\nu}$ at that point, where $\nu$ is the order of the cyclic subgroup associated to the elliptic fixed point. On the other hand, if we pick local coordinates at a cusp, we will find $f(z) \sim \log z$ near that cusp; thus the Riemann surface $X$ has $p$ infinitely long tubes and the cusps can be thought of as the ends of these tubes at infinity. Intuitively, the "boundary" of $X$

should be identified with the set of cusps, in agreement with our construction of the Kleinian boundary since in this example $D_\Gamma = \emptyset$. This discretization was predicted earlier by the holographic argument.

As another example, we consider some torsion-free discrete subgroups of $\mathrm{PSL}(2,\mathbb{C})$ acting on $\mathbf{H}^3$ and on the boundary sphere $\mathbf{S}^2 \cong \hat{\mathbb{C}} \equiv \mathbb{C} \cup \{\infty\}$. Let us begin with a $\Gamma$ for which the limit set $\Lambda_\Gamma$ constitutes a circle $\mathbf{S}^1$ in the extended complex plane $\hat{\mathbf{C}}$ [17]. In this case, the domain of discontinuity $D_\Gamma = D_\Gamma^+ \cup D_\Gamma^-$ has two connected components (disks) $D_\Gamma^\pm$, separated from each other by that circle. Since $\Gamma$ acts properly discontinuously on $D_\Gamma$, the quotient space $M' = D_\Gamma/\Gamma$ is a surface with two components $M'^\pm = D_\Gamma^\pm/\Gamma$, which inherits a comformal structure from $D_\Gamma$. Intuitively, this surface gives the main body of the "boundary" of $X = \mathbf{H}^3/\Gamma$ and in fact it is the continuous part of the kleinian boundary. The volume of $X$ is infinite since $M'$ has nonzero area.

Now we will focus on what can happen when $\Gamma$ is deformed into a limiting group $\Gamma^*$ so that $\mathrm{Vol}(\mathbf{H}^3/\Gamma^*) < \infty$. Suppose that $\Gamma^*$ arises as the fundamental group of a certain Riemann surface $M^*$. According to [19], under a quasi-conformal deformation of $\Gamma$ the limit set $\Lambda_\Gamma$ is also deformed, from the circle $\mathbf{S}^1$ to a Jordan curve in $\hat{\mathbf{C}}$, and the conformal structures $j^\pm$ of $M'^\pm$ departure from their original positions in Teichmuller space. The limiting group $\Gamma^*$ can be obtained by moving $j^\pm$ simultaneously towards two boundary points $j^\pm(\infty)$ of Teichmuller space. One of the boundary points, say $j^+(\infty)$, is determined by pinching a lamination $\lambda^+$ of $M^*$ and, if we pick another lamination $\lambda^-$ such that $\lambda^\pm$ fill up $M^*$, then by pinching $\lambda^-$ we can get the other boundary point, $j^-(\infty)$. As $j^\pm$ approach $j^\pm(\infty)$, the Jordan curve $\Lambda_\Gamma$ becomes more and more complicated, — it can eventually fill the whole complex plane when the double limit is reached [19]. The hyperbolic structure of the final quotient space $X^* = \mathbf{H}^3/\Gamma^*$ is typically described by Thurston's mapping torus constructed from identifying $(M^*, 1)$ to $(\phi(M^*), 0)$ in the cylinder $M^* \times [0,1]$, where $\phi : M^* \to M^*$ is a diffeomorphism and when it is regarded as a transformation of Teichmuller space, it fixes precisely the two boundary points $j^\pm(\infty)$. Obviously, the resulting mapping torus has no boundary if $M^*$ is compact. This can be expected since $D_{\Gamma^*} = \emptyset$ and $\Gamma^*$ has no cusps for a compact $M^*$. Cusps of $\Gamma^*$ can arise if $M^*$ is not compact. The set of all such cusps should now be identified with the Kleinian boundary of $X^*$, and thus we have a concrete model to see how $M'$ can be discretized during the limiting process $\Gamma \to \Gamma^*$. Again, these geometrical considerations are consistent with the holographic prediction given at the begining of this section: finite-volume bulk space has a discrete kleinian boundary.

Before leaving these examples let us consider the cusped manifolds [18, 17, 19] in some detail. Suppose that $X$ is a $(d+1)$-dimensional hyperbolic manifold in either of the above examples (thus $d = 1, 2$), having finite volume $\mathrm{Vol}(X)$ and containing $p$

cusps. Mathematically, there is a lower bound for the hyperbolic volume:

$$\mathrm{Vol}(X) \geq c_d \cdot p\,, \tag{3.7}$$

where $c_d$ is some constant of order 1, which may depend on what the example we are considering, — in the 2-dimensional example we have [18] $c_1 = 2\pi$ and, in the 3-dimensional example [21], $c_2 \approx 1.01494$. According to our geometrical construction, the Kleinian boundary $M'$ of $X$ should be discretized completely in this finite volume case, consisting of the $p$ cusps. The average spacing of points in such a "lattice boundary space" should behave as

$$\Delta x_{\parallel} \sim p^{-1/d}\,. \tag{3.8}$$

Note that (3.8) also has an interpretation as the uncertainty of localizing a spacetime point in the boundary theory. Going to the bulk manifold, the local geometry of $X$ reads

$$ds^2 = \frac{U^2}{R^2}\,dx_{\parallel}^2 + \frac{R^2}{U^2}\,dU^2\,, \tag{3.9}$$

so the volume of $X$ may be estimated roughly

$$\mathrm{Vol}(X) \sim \int_0^{\Delta U} dU\,U^{d-1} \sim (\Delta U)^d\,, \tag{3.10}$$

with $\Delta U$ being an effective infrared cutoff of the bulk manifold, which can be identified with the length scale of a fundamental domain $\mathcal{F} \subset \mathbf{H}^{d+1}$ for $\Gamma$. Now we combine (3.7), (3.8) and (3.10) to derive

$$\Delta x_{\parallel} \cdot \Delta U \geq \sqrt[d]{c_d} \sim 1\,. \tag{3.11}$$

One recognizes immediately that (3.11) is the UV/IR relation in string units, reformulated as a kind of the space-time uncertainty principle [20]. This gives us further evidence that the geometrical definition of $M'$ can serve as the space on which the holographic boundary theory lives. The discussion also suggests that cusps are capable of storing physical information. In appendix B, we will give an estimation of information density in $d = 4$ dimensions.

## 4. Conformal fields on the boundary

Having established a mathematical description for the Kleinian boundary of $X$, we shall try to explore some unusual behavior of the boundary conformal fields. In particular we want to show how the physical spectrum can depend nontrivially on a generic discrete group $\Gamma$. Here, for simplicity, we will only consider conformal fields without Lorentz indices; our discussion can be easily generalized to the case containing several Lorentz quantum numbers.



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We begin with conformal transformations of scalar fields on the Kleinian boundar $M'$. For this discussion we need only to consider the continuous part $D_\Gamma/\Gamma$ of $M'$, ignoring all cusps. So given any $f$ in the conformal group (2.4), a scalar field $\mathcal{O}(x)$ of conformal dimension $\Delta$ will transform as

$$\mathcal{O}(x) \longrightarrow \mathcal{O}^f(x) \equiv \mathcal{U}_f^{-1}\,\mathcal{O}(x)\,\mathcal{U}_f = |f'(x)|^{\Delta}\,\mathcal{O}(fx) \tag{4.1}$$

where $\mathcal{U}$ is some unitary representation of the conformal group and $|f'(x)|$ is the Euclidean distortion, defined through $|fx - fy|^2 = |f'(x)| \cdot |f'(y)| \cdot |x-y|^2$ for any $x, y \in \hat{\mathbf{R}}^d$.

If $\Delta \neq 0$, one cannot really think of $\mathcal{O}(x)$ as an operator-valued distribution since it depends on the choice of a Riemannian metric on $D_\Gamma/\Gamma$. In particular under Weyl transformation

$$h_{ij} \to \tilde{h}_{ij} = e^{2w} \cdot h_{ij} \tag{4.2}$$

the field $\mathcal{O}(x)$ will get rescaled, $\mathcal{O} \to \widetilde{\mathcal{O}} = e^{-\Delta w} \cdot \mathcal{O}$. In certain applications one may wish to find an intrinsic description for conformal fields in the sense that it does not depend on the choice of Riemannian metrics. For this purpose, we will invoke the concept of "conformal densities" considered in [22]. Recall that [22] a conformal density of dimension $\Delta$ is some positive finite measure $\mu$ assigned to the given metric $h_{ij}$, and under the Weyl rescaling (4.2), it obeys a transformation law[5]

$$d\mu \to d\tilde{\mu} = e^{\Delta w} \cdot d\mu\,. \tag{4.3}$$

Thus, by taking a measurable subset $A$ of $D_\Gamma/\Gamma$, one can form an integral invariant under Weyl transformations:

$$\mathcal{O}(A) \equiv \int_A \mathcal{O}(x)\, d\mu(x)\,, \qquad A \subset D_\Gamma/\Gamma\,. \tag{4.4}$$

This clearly gives the desired intrinsic discription for $\mathcal{O}(x)$. Note that the existence of such a description is essential for formulating the explicit $AdS/CFT$ correspondence [2, 3]. In that connection, the conformal density $d\mu(x)$ will serve as a source coupled to the local field $\mathcal{O}(x)$ on the boundary. The coupling should be conformally invariant by physical requirements.

One can lift fields from the quotient manifold $D_\Gamma/\Gamma$ to its covering space $D_\Gamma$ and regard $\mathcal{O}(x)$ as a conformal field $\bar{\mathcal{O}}(x)$ on $D_\Gamma$, obeying the $\Gamma$-invariant condition $U_\gamma^{-1}\bar{\mathcal{O}}(x)U_\gamma = \bar{\mathcal{O}}(x)$, or, equivalently:

$$\bar{\mathcal{O}}(\gamma x) = |\gamma'(x)|^{-\Delta} \cdot \bar{\mathcal{O}}(x)\,, \qquad \forall \gamma \in \Gamma\,. \tag{4.5}$$

[5]Mathematically, this transformation law should be understood as a constraint on the two assignments $h_{ij} \to \mu$, $\tilde{h}_{ij} \to \tilde{\mu}$ related by $\tilde{h}_{ij} = e^{2w} \cdot h_{ij}$, and this constraint will force the value of the Radon-Nikodym derivative $\frac{d\tilde{\mu}}{d\mu}$ to be $e^{\Delta w}$.



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To lift (4.4), let $\bar{A}$ denote the covering space of $A$ and suppose that $\mathcal{F} \subset D_\Gamma$ is a fundamental domain for $\Gamma$. Since the images of $\mathcal{F}$ under the action of $\Gamma$ tesselate $D_\Gamma$, we have

$$\begin{aligned}\bar{\mathcal{O}}(\bar{A}) \equiv \int_{\bar{A}} \bar{\mathcal{O}}(x)\, d\bar{\mu}(x) \;&=\; \sum_{\gamma\in\Gamma} \int_{\gamma(\mathcal{F}\cap\bar{A})} \bar{\mathcal{O}}(x)\, d\bar{\mu}(x) \\ &= \sum_{\gamma\in\Gamma} \int_{\mathcal{F}\cap\bar{A}} \bar{\mathcal{O}}(\gamma x)\, d\bar{\mu}(\gamma x) = \int_{\mathcal{F}\cap\bar{A}} \bar{\mathcal{O}}(x)\, d\nu(x)\,,\end{aligned} \tag{4.6}$$

where $\bar{\mu}$ is a conformal density on $D_\Gamma$, of dimension $\Delta$, and

$$d\nu(x) = \sum_{\gamma\in\Gamma} |\gamma'(x)|^{-\Delta} \cdot d\bar{\mu}(\gamma x) \tag{4.7}$$

stands for the $\Gamma$-invariant projection of $d\bar{\mu}(x)$, which may be identified with a conformal density on $D_\Gamma/\Gamma$ having the same dimension as $d\bar{\mu}(x)$. Hence the last integral in (4.6) can be regarded as the integral (4.4) under the identifications $\mathcal{F}\cap\bar{A} \leftrightarrow A$ and $d\nu \leftrightarrow d\mu$.

As $D_\Gamma$ constitutes a subset of $\mathbf{S}^d$, the measure $d\bar{\mu}(x)$ has a natural extension to the whole boundary sphere; hence the projection (4.7) defines a $\Gamma$-invariant conformal density on $\mathbf{S}^d$. Now for generic $\Gamma$, the existence of such an invariant measure will impose a constraint on the allowed values of $\Delta$. In fact, if $\Gamma$ is a non-elementary discrete group, then according to Corollary 4 of [22], the dimension of any conformal density on $\mathbf{S}^d$ invariant under $\Gamma$ has a lower bound. This bound can be saturated and agrees with the critical exponent $\delta(\Gamma)$ of $\Gamma$, which is defined so that the Poincaré series

$$g(x,y;s) = \sum_{\gamma\in\Gamma} \exp(-s\,\rho(x,\gamma y)) \qquad (x,y\in\mathbf{H}^{d+1}\,,\ s\in\mathbf{R}) \tag{4.8}$$

converges for $s > \delta(\Gamma)$ and diverges for $s < \delta(\Gamma)$, here $\rho(\cdot,\cdot)$ is the hyperbolic distance. As a consequence, scalar conformal fields on $M'$ are not intrinsically defined in the sense of (4.4) unless their conformal dimensions obey the inequality

$$\Delta \geq \delta(\Gamma)\,. \tag{4.9}$$

If there are no stronger constraints presented, $\Delta$ can saturate the lower bound in (4.9), so in this case the conformal dimension of the most relevant operators on $M'$ is $\delta(\Gamma)$.

The above argument rests on the requirement that all conformal fields on $D_\Gamma/\Gamma$ should have an intrinsic description. However, there exists other evidence supporting the constraint (4.9). Consider, for example, a string theory in $\mathbf{H}^{d+1}\times W$ which corresponds to some CFT on the boundary of $\mathbf{H}^{d+1}$. With a suitable choice of $W$ we may assume this string theory contains zero-branes. Each zero-brane on the



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boundary sphere $\mathbf{S}^d$ with mass $m$ is associated to a local conformal field $\mathcal{O}(x)$, whose dimension is determined by $\Delta = (d + \sqrt{d^2 + 4m^2})/2$. The two-point Green's function $G(x, y) \equiv \langle \mathcal{O}(x)\mathcal{O}^\dagger(y)\rangle$ can be computed within the bulk theory: For $\Delta \sim m \gg 1$, we have

$$G(x, y) \sim \exp(-m\,\rho(x, y)) \sim \exp(-\Delta\,\rho(x, y))\,. \tag{4.10}$$

Note that (4.10) should be regularized by moving $x, y$ "slightly" from the boundary of $\mathbf{H}^{d+1}$ to its interior. With this regularization the Poincaré series $G_\Gamma(x, y) \equiv \sum_{\gamma\in\Gamma} G(x, \gamma y)$ is biautomorphic, that is, $G_\Gamma(\gamma_1 x, \gamma_2 y) = G_\Gamma(x, y)$ for any $\gamma_1, \gamma_2 \in \Gamma$. (This follows from the fact that $\Gamma$ acts isometrically on the hyperbolic space, so in particular we have $\rho(\gamma x, \gamma y) = \rho(x, y)$, $\forall \gamma \in \Gamma$.) It is therefore tampting to interprete $G_\Gamma(x, y)$ as a regularized two-point Green's function of certain CFT defined on the quotient manifold $M' \subset \mathbf{S}^d/\Gamma$. Evidently, this interpretation can work only if $G_\Gamma(x, y)$ converges. When $G_\Gamma(x, y) < \infty$, it is possible to construct a local conformal field $\mathcal{O}_\Gamma(x)$ on $M'$ such that $< \mathcal{O}_\Gamma(x)\mathcal{O}_\Gamma^\dagger(y) >$ is regularized to be $G_\Gamma(x, y)$, by a generalized Osterwalder-Schrader quantization procedure. The conformal dimension $\Delta$ of such an $\mathcal{O}_\Gamma(x)$ should then obey (4.9). Actually, from (4.8) and (4.10), one easily derives

$$G_\Gamma(x, y) \sim g(x, y;\, \Delta)\,, \tag{4.11}$$

so convergence of $G_\Gamma(x, y)$ requires the inequality $\Delta \geq \delta(\Gamma)$.

So far we have derived a general constraint (4.9) on the physical spectrum of conformal fields living in $M'$. While this inequality is always presented for conformal fields having the intrinsic description (4.4), it may not really provide a useful constraint in the $AdS/CFT$ correspondence. As a matter of fact, if a scalar conformal field arises from the $AdS_{d+1}/CFT_d$ correspondence, then its conformal dimension should obey another inequality [3]

$$\Delta \geq \frac{d}{2}\,. \tag{4.12}$$

Thus, if $\delta(\Gamma) \leq \frac{d}{2}$, the constraint (4.9) is weaker than (4.12) and it will be automatically satisfied in the boundary CFT. In this case our constraint has no observable effect on the underlying physical spectrum. In order for (4.9) to have visible effect, we have to consider another case, $\delta(\Gamma) > \frac{d}{2}$, where taking the quotient by $\Gamma$ will change the boundary theory drastically. Note that the case $\delta(\Gamma) > \frac{d}{2}$ is also interesting from the mathematical point of view [22].

The above results can be rederived in the bulk theory via the quotient $AdS/CFT$ correspondence. Consider a scalar field $\phi(y)$ with mass $m$ in the bulk spacetime $X = \mathbf{H}^{d+1}/\Gamma$. The classical action is given by

$$I(\phi) = \frac{1}{2}\int_X d^{d+1}y\,\sqrt{g}\,(g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi + m^2\phi^2)\,. \tag{4.13}$$

The Klein-Gordon equation for $\phi$ has the form

$$\nabla^2\phi = \lambda\phi\,, \tag{4.14}$$

with $\nabla^2 \equiv \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu)$ and $\lambda \equiv m^2$, whose solutions are known as $\lambda$-harmonic functions. There is a real number

$$\lambda_0(X) = -\inf_\phi \frac{\int_X |\nabla\phi|^2}{\int_X |\phi|^2} \leq 0 \tag{4.15}$$

separating the $L^2$-spectrum of $\nabla^2$ (which is contained in the region $-\infty < \lambda \leq \lambda_0(X)$) from the "positive" spectrum (which is contained in the region $\lambda_0(X) \leq \lambda < \infty$), and for generic $\Gamma$ this number takes the value [23]

$$\lambda_0(X) = \begin{cases} -\frac{1}{4}d^2, & \text{if } \delta(\Gamma) \leq \frac{1}{2}d, \\ \delta(\Gamma)(\delta(\Gamma) - d), & \text{if } \delta(\Gamma) \geq \frac{1}{2}d. \end{cases} \tag{4.16}$$

Now, given any boundary values $\phi_0(\mathbf{x})$ for $\mathbf{x} \in M'$ and a positive function $f(y)$ on $\bar{X} \equiv X \cup M'$ which has a first-order zero on the boundary, one wants to extend $\phi_0(\mathbf{x})$ uniquely to a $\lambda$-harmonic function $\phi$ in the interior of $X$, such that $\phi(y) \to f^{(d-\sqrt{d^2+4\lambda})/2} \cdot \phi_0$ as $y \in X$ approaches a boundary point. Stability requires that we should only consider positive $\lambda$-harmonic extensions, namely solutions of (4.14) with $\lambda \equiv m^2 \geq \lambda_0(X)$, and all the $L^2$-solutions with $\lambda$ strictly less than $\lambda_0(X)$ should be discarded (since they are precisely the normalizable zero modes that will lead to instabilities). According to the $AdS/CFT$ correspondence, the conformal operator $\mathcal{O}(\mathbf{x})$ coupled naturally to the source $\phi_0(\mathbf{x})$ on the quotient boundary has the scaling dimension

$$\Delta = \frac{d + \sqrt{d^2 + 4m^2}}{2} \geq \frac{d + \sqrt{d^2 + 4\lambda_0(X)}}{2}\,. \tag{4.17}$$

Thus, combining this and (4.16) we see that $\Delta \geq d/2$ if $\delta(\Gamma) \leq d/2$ and $\Delta \geq \delta(\Gamma)$ if $\delta(\Gamma) \geq d/2$, in ageement with the previous argument.

To determine the dependence of $\delta(\Gamma)$ on $\Gamma$, let us recall a geometrical characteristic of the critical exponent [22]. Suppose first that $\Gamma$ is a convex cocompact group, namely, $\Gamma$ has a fundamental domain in $\mathbf{H}^{d+1}$ which has finitely many sides and does not touch the limit set $\Lambda_\Gamma$. Sometimes such groups are also called "geometrically finite without cusps" and they form quite a rich class in hyperbolic geometry. If $\Gamma$ is convex cocompact, then it can be shown ([22], section 3) that the critical exponent of $\Gamma$ agrees exactly with the Hausdorff dimension of the limit set:

$$\delta(\Gamma) = d_H(\Lambda_\Gamma)\,. \tag{4.18}$$

Next, for a more general non-elementary discrete group $\Gamma$, its critical exponent turns out to be greater than or equal to the Hausdorff dimension of the radial limit set $\Lambda_\Gamma^{rad}$. Recall that [22] a point $x$ of $\mathbf{S}^d$ belongs to $\Lambda_\Gamma^{rad}$ if and only if there are infinitely many $y_1, y_2, \ldots \in \mathbf{H}^{d+1}$ in some orbit $\Gamma \cdot y$, such that the distance between $y_n$ and a geodesic ray ending at $x$ is bounded for all $n$. Using this definition one easily checks that $\Lambda_\Gamma^{rad} \subset \Lambda_\Gamma$. It is conjectured that not only the inequality $\delta(\Gamma) \geq d_H(\Lambda_\Gamma^{rad})$ holds, but actually one has the equality

$$\delta(\Gamma) = d_H(\Lambda_\Gamma^{rad})\,, \tag{4.19}$$

and this conjecture was verified when $\Gamma$ has a $\delta(\Gamma)$-finite volume ([22], section 6), including all (finitely generated) Fuchian groups. Note that (4.19) implies (4.18) in the convex cocompact case since in that case we have $\Lambda_\Gamma^{rad} = \Lambda_\Gamma$.

From the foregoing discussion we expect that $\delta(\Gamma)$ may increase as $\Gamma$ becomes larger and larger. In particular when $\Lambda_\Gamma^{rad} = \Lambda_\Gamma$ fills the whole boundary sphere $\mathbf{S}^d$, or the Hausdorff dimension $d_H(\Lambda_\Gamma^{rad})$ reaches its maximal value $d$, then according to the constraint (4.9), there will be no relevant operators in the boundary theory. As we have seen in the previous section, this can occur if $\Gamma$ becomes so large that the volume of $X$ is finite; in such an extreme case we have $m^2 \geq \lambda_0(X) = 0$ for all scalar fields in the bulk and thus no tachyon modes can exist. It would be very interesting to understand the underlying bulk/boundary correspondence in terms of brane dynamics (cf. [24]).

## 5. Computing the scalar 2-point function

In this section we shall compute the scalar two-point function explicitly using the quotient $AdS/CFT$ correspondence. The computation is a $\Gamma$-invariant version of that presented in [3]. For simplicity, we shall assume that $\Gamma$ has no cusps, or at least the cusps do not meet the limit set $\Lambda_\Gamma$, so that the Kleinian boundary of $X = \mathbf{H}^{d+1}/\Gamma$ is simply given by $M' = D_\Gamma/\Gamma$.

Following [3], let us solve the Klein-Gordon equation (4.14) defined in $X$ with the given boundary values $\phi_0(\mathbf{x})$, $\mathbf{x} \in M'$. For this purpose we need to construct a (generalized) Poisson kernel $K(y, \mathbf{x})$ on $X \times M'$ such that $(\nabla_y^2 - m^2)K(y, \mathbf{x}) = 0$ for all $y$ in $X$ and

$$\phi(y) = \int_{M'} d^d\mathbf{x}\, K(y, \mathbf{x}) \cdot \phi_0(\mathbf{x}) \tag{5.1}$$

has the correct boundary behavior. On the covering space $\mathbf{H}^{d+1} \times D_\Gamma$ of $X \times M'$, the Poisson kernel $k(y, \mathbf{x})$ is known to be [3]

$$k(y, \mathbf{x}) = const. \times \frac{y_0^\Delta}{(y_0^2 + |\mathbf{y} - \mathbf{x}|^2)^\Delta}\,, \tag{5.2}$$

where $\Delta$ is defined in (4.17). However, this expression does not meet our requirement of $\Gamma$-invariance. In fact, given any $\gamma \in \Gamma$, we have $\rho(\gamma y, \gamma x) = \rho(y, x)$ and

$\cosh\rho(y,x) = 1 + \frac{|y-x|^2}{2y_0x_0}$, so that

$$k(\gamma y, \gamma \mathbf{x}) = \left(\lim_{x_0\to 0}\frac{x_0}{\gamma(x)_0}\right)^{\Delta} \cdot k(y,\mathbf{x}) = |\gamma'(\mathbf{x})|^{-\Delta}\cdot k(y,\mathbf{x}). \tag{5.3}$$

Thus in general we do not have $k(\gamma y,\mathbf{x}) = k(y,\mathbf{x})$. To find the $\Gamma$-invariant Poisson kernel, we will invoke the familiar Poincaré series

$$K(y,\mathbf{x}) \equiv \sum_{\gamma\in\Gamma} |\gamma'(\mathbf{x})|^{\Delta}\cdot k(y,\gamma\mathbf{x})\,, \qquad y\in\mathbf{H}^{d+1}\,, \quad \mathbf{x}\in D_\Gamma\,. \tag{5.4}$$

It is easy to check that (i) $K(y,\mathbf{x})$ has the desired $\Gamma$-invariant propertiy $K(\gamma y,\mathbf{x}) = K(y,\mathbf{x})$ for any $\gamma\in\Gamma$, (ii) $K(y,\mathbf{x})$ transforms covariantly under $\mathbf{x}\to\gamma\mathbf{x}$, namely $K(y,\gamma\mathbf{x}) = |\gamma'(\mathbf{x})|^{-\Delta}\cdot K(y,\mathbf{x})$, and (iii) $K(y,\mathbf{x})$ solves the Klein-Gordon equation in $\mathbf{H}^{d+1}$. Thus, combining (i) and (iii) we see that (5.4) is in fact a solution of the Klein-Gordon equation in $X$. Before we can identify such $K(y,\mathbf{x})$ with the Poisson kernel defined on $X\times M'$, we need to check further that substituting (5.4) into (5.1) will lead to the correct boundary behavior of $\phi(y)$.

To this end we take a $\Gamma$-invariant conformal density (in the sense of [22]) of dimension $\Delta$, defined on the covering space $D_\Gamma$ of the quotient boundary:

$$d\bar\mu(\mathbf{x}) \equiv \bar\phi_0(\mathbf{x})\, d^d\mathbf{x}\,, \qquad d\bar\mu(\gamma\mathbf{x}) = |\gamma'(\mathbf{x})|^{\Delta}\cdot d\bar\mu(\mathbf{x})\,. \tag{5.5}$$

One requires that the function $\bar\phi_0(\mathbf{x})$ in equation (5.5) should transform as $\bar\phi_0(\gamma\mathbf{x}) = |\gamma'(\mathbf{x})|^{\Delta-d}\cdot\bar\phi_0(\mathbf{x})$ under $\gamma\in\Gamma: D_\Gamma\to D_\Gamma$, namely, it should look like a field of conformal dimension $d-\Delta$. In particular we can take $\bar\phi_0(\mathbf{x})$ to be the lifting of the boundary values $\phi_0(\mathbf{x})$ from $M'$ to its covering space $D_\Gamma$, which has the correct conformal dimension. With this choice we pick an arbitrary fundamental domain $\mathcal{F}\subset D_\Gamma$ of $\Gamma$ and then consider

$$\begin{aligned}\int_{D_\Gamma} d^d\mathbf{x}\, k(y,\mathbf{x})\,\bar\phi_0(\mathbf{x}) &= \sum_{\gamma\in\Gamma}\int_{\gamma(\mathcal{F})} k(y,\mathbf{x})\, d\bar\mu(\mathbf{x}) = \sum_{\gamma\in\Gamma}\int_{\mathcal{F}} k(y,\gamma\mathbf{x})\,|\gamma'(\mathbf{x})|^{\Delta}\, d\bar\mu(\mathbf{x})\\ &= \int_{\mathcal{F}} d^d\mathbf{x}\, K(y,\mathbf{x})\,\bar\phi_0(\mathbf{x}) = \int_{M'} d^d\mathbf{x}\, K(y,\mathbf{x})\,\phi_0(\mathbf{x})\,.\end{aligned} \tag{5.6}$$

Notice that the first integral in (5.6) has the correct boundary behavior, since $k(y,\mathbf{x})$ is the Poisson kernel on the covering space. The last integral thus gives the desired solution of the Klein-Gordon equation in $X$ with the boundary values $\phi_0(\mathbf{x})$.

The above discussion also explains why the boundary theory dual to supergravity on $X$ should live exactly on $M'$. If it is defined on a subspace $N$ of $M'$ with $M'-N\neq\emptyset$, then we can cover $N$ by a subspace $E$ of $D_\Gamma$ such that $D_\Gamma - E$ is non-empty and invariant under $\Gamma$. We may thus replace $\mathcal{F}$ in (5.6) by $\mathcal{F}\cap E$ to derive

$$\int_E d^d\mathbf{x}\, k(y,\mathbf{x})\,\bar\phi_0(\mathbf{x}) = \int_N d^d\mathbf{x}\, K(y,\mathbf{x})\,\phi_0(\mathbf{x}) \equiv \phi(y), \tag{5.7}$$

where $\phi(y)$ solves the equations of motion in the bulk. However, this solution has a rather special asymptotic behavior as $y$ in the l.h.s. of (5.7) approaches $D_\Gamma - E$, and the "boundary value" $\bar{\phi}_0(y) \equiv y_0^{\Delta-d}\phi(y)$ vanishes at any $y = \mathbf{y} \in D_\Gamma - E$ due to the known behavior of the Poinsson kernel on the covering space [3], $k(\mathbf{y}, \mathbf{x}) \propto y_0^{d-\Delta}\delta^d(\mathbf{x}-\mathbf{y})$. Thus (5.7) cannot describe more general solutions with the asymptotic behavior $\lim_{y\to\mathbf{y}} \phi_0(y) \neq 0$ for $\mathbf{y} \in (D_\Gamma - E)/\Gamma$, which is of course allowed in the quotient bulk theory. It follows that bulk supergravity on $X$ cannot be faithfully represented by a theory on $N \subset M'$, unless $N = M'$. On the other hand, if the boundary theory is defined on a space $N$ containing $M'$ with $N - M' \neq \emptyset$, then the covering space $E$ of $N$ must consist of $D_\Gamma$ together with some points (not cusps) in the limit set $\Lambda_\Gamma$. When this happens, Eq.(5.7) will no longer hold, since $\Gamma$ now does not act properly discontinuously on $E$ and accordingly we cannot decompose $E$ canonically into fundamental domains $\mathcal{F} \cap E$. To construct a solution of the Klein-Gordon equation in $X$ using boundary data on $N = E/\Gamma$, the best we can do is to formulate the integral (5.6) over $D_\Gamma$. One may then ask whether the $\Gamma$-invariant boundary data $\bar{\phi}_0(\mathbf{x})$ for $\mathbf{x}$ outside $D_\Gamma$ is redundant. The answer is affirmative because each $\mathbf{x} \in E - D_\Gamma$ is determined by $\mathbf{x} = \lim_{n\to\infty} \gamma_n(\mathbf{x}')$ with $\mathbf{x}' \in D_\Gamma$ and $\gamma_n \in \Gamma$, so as long as the boundary values of $\phi$ on $D_\Gamma/\Gamma$ are given, we need not to specify the values of $\phi$ separately on $(E - D_\Gamma)/\Gamma$. In other words, the space $M' = D_\Gamma/\Gamma$ is sufficient to define the boundary conditions for solving the Klein-Gordon equation in $X$.

We are now ready to evaluate the action (4.13). Let us choose a fundamental domain $\mathcal{G}$ of $\Gamma$ in $\mathbf{H}^{d+1}$ such that $\bar{\mathcal{G}} \cap \mathbf{S}^d = \bar{\mathcal{G}} \cap D_\Gamma = \mathcal{F}$. This is possible since we have assumed that all possible cusps of $\Gamma$ do not live in the limit set $\Lambda_\Gamma$. We can represent the integral in (4.13) over $X$ by a corresponding integral over $\mathcal{G}$ and, by integrating by parts, one finds that the action $I(\phi)$ consists of two terms, one of which vanishes due to the equations of motion and the other of which can be reduced to a surface integral [3]

$$I(\phi) = \frac{1}{2}\int_{\mathcal{G}} d^{d+1}y\, \partial_\mu \left(\sqrt{g}\, g^{\mu\nu}\, \phi\, \partial_\nu \phi\right) = \lim_{\epsilon\to 0}\, \frac{1}{2}\int_{\bar{\mathcal{G}}\cap T_\epsilon} d^d\mathbf{y}\, \sqrt{h}\, \phi\, \mathbf{n}\cdot\nabla\, \phi\,, \tag{5.8}$$

where $T_\epsilon$ is the horosphere $y_0 = \epsilon$ based at $\infty$, $h_{ij} = \delta_{ij}/\epsilon^2$ is the metric induced from $g_{\mu\nu}$, and $\mathbf{n}\cdot\nabla = y_0 \frac{\partial}{\partial y_0}|_{y_0=\epsilon}$ is the derivative along the direction normal to the suface $\bar{\mathcal{G}} \cap T_\epsilon$ in $\mathcal{G}$. Now using (5.1), (5.2), (5.4) and the fact $\lim_{\epsilon\to 0} \bar{\mathcal{G}} \cap T_\epsilon = \mathcal{F}$, one gets

$$I(\phi) = const. \times \int_{\mathcal{F}\times\mathcal{F}} d^d\mathbf{x} d^d\mathbf{y}\, \bar{\phi}_0(\mathbf{x}) \left(\sum_{\gamma\in\Gamma} \frac{|\gamma'(\mathbf{y})|^\Delta}{|\mathbf{x} - \gamma\mathbf{y}|^{2\Delta}}\right) \bar{\phi}_0(\mathbf{y})\,, \tag{5.9}$$

with $\bar{\phi}_0(\mathbf{x})$ being the lifting of $\phi_0(\mathbf{x})$.

Since our computation manifests $\Gamma$-invariance, the above expression is actually an integral over $M' \times M'$. This may be seen directly from (5.9). For this purpose, we take two Möbius transformations $\mathbf{x} \to \gamma_1 \mathbf{x}$, $\mathbf{y} \to \gamma_2 \mathbf{y}$. Using (5.5) as well as the distortion formula

$$\frac{|\gamma'(\gamma_2\mathbf{y})|^{\Delta}}{|\gamma_1\mathbf{x} - \gamma\gamma_2\mathbf{y}|^{2\Delta}} = \frac{|\gamma_1'(\mathbf{x})|^{-\Delta} \cdot |(\gamma_1^{-1}\gamma\gamma_2)'(\mathbf{y})|^{\Delta} \cdot |\gamma_2'(\mathbf{y})|^{-\Delta}}{|\mathbf{x} - \gamma_1^{-1}\gamma\gamma_2\mathbf{y}|^{2\Delta}}\,, \tag{5.10}$$

(which may be verified directly from the definition), we find that the integral (5.9) is invariant if we replace its integration domain $\mathcal{F} \times \mathcal{F}$ by $\gamma_1(\mathcal{F}) \times \gamma_2(\mathcal{F})$ and, at the same time, replace $\Gamma$ by $\tilde{\Gamma} \equiv \gamma_1 \Gamma \gamma_2^{-1}$. Thus, if $\gamma_1, \gamma_2 \in \Gamma$, then $\tilde{\Gamma}$ coincides exactly with $\Gamma$ and $\gamma_{1,2}(\mathcal{F})$ constitute fundamental domains. Consequently, (5.9) is independent of the choice of fundamental domains in both factors of $\mathcal{F} \times \mathcal{F}$, hence giving rise to an integral of the form

$$I(\phi) = \int_{M' \times M'} d^d\mathbf{x} d^d\mathbf{y}\, \phi_0(\mathbf{x})\, G(\mathbf{x}, \mathbf{y})\, \phi_0(\mathbf{y})\,, \qquad G(\mathbf{x}, \mathbf{y}) \propto \sum_{\gamma \in \Gamma} \frac{|\gamma'(\mathbf{y})|^{\Delta}}{|\mathbf{x} - \gamma\mathbf{y}|^{2\Delta}}\,. \tag{5.11}$$

According to the quotient version of the $AdS/CFT$ correspondence, the function $G(\mathbf{x}, \mathbf{y})$ in the bulk action (5.11) is to be identified with the two-point correlator $< \mathcal{O}(\mathbf{x})\mathcal{O}(\mathbf{y}) >$ of the boundary theory, where $\mathcal{O}(\mathbf{x})$ is a scalar field on $M'$ of conformal dimension $\Delta$. We thus expect that $G(\mathbf{x}, \mathbf{y})$ should be consistent with conformal invaiance of the boundary theory; in particular, it should transform correctly under the unbroken conformal group (2.4) on $M'$, namely for any $f \in \mathrm{SO}(d+1, 1)$ such that $f^{-1}\Gamma f = \Gamma$, we should have $G(f(\mathbf{x}), f(\mathbf{y})) = |f'(\mathbf{x})|^{-\Delta} |f'(\mathbf{y})|^{-\Delta} G(\mathbf{x}, \mathbf{y})$. As a quick check, taking $\gamma_1 = \gamma_2 = f$ in (5.10) to derive

$$\sum_{\gamma \in \Gamma} \frac{|\gamma'(f(\mathbf{y}))|^{\Delta}}{|f(\mathbf{x}) - \gamma f(\mathbf{y})|^{2\Delta}} = |f'(\mathbf{x})|^{-\Delta} |f'(\mathbf{y})|^{-\Delta} \sum_{\gamma \in f^{-1}\Gamma f} \frac{|\gamma'(\mathbf{y})|^{\Delta}}{|\mathbf{x} - \gamma\mathbf{y}|^{2\Delta}}\,. \tag{5.12}$$

Hence $G(\mathbf{x}, \mathbf{y})$ indeed has the desired transformation properties under (2.4).

The above computation has several straightforward extensions. First we can consider "twisted sectors" by incorporating a factor $\chi(\gamma)$ in each term of the Poincaré series (5.4), where $\chi$ is some one-dimensional unitary representation of the discrete group $\Gamma$. Secondly, the computation can be extended directly to correlation functions involving other local fields in the boundary theory, such as fields with Lorentz and internal indices. The general form of two-point functions should look like a Poincaré series which arises from taking the $\Gamma$-invariant projection. It is also possible to compute higher-point functions by investigating nontrivial interactions in the supergravity action.

## 6. Conclusions

In this paper we have made several observations supporting a quotient version of the $AdS/CFT$ correspondence. One observation is that given bulk spacetime of the form



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$X = \mathbf{H}^{d+1}/\Gamma$, the quantized conformal field theory associated to tree-level supergravity in $X$ should live on the Kleinian boundary $M' = (D_\Gamma/\Gamma) \cup \{\textit{cusps not in } D_\Gamma\}$ of $X$, rather than on the naive quotient space $M = \mathbf{S}^d/\Gamma$. This is consistent both with the holographic principle and with a geometrical consideration. In particular, we explained how this Kleinian boundary manifold can be completely discretized when the domain of discontinuity $D_\Gamma$ collapses or, equivalently, the limit set $\Lambda_\Gamma$ fills the whole boundary sphere $\mathbf{S}^d$ of the hyperbolic space, with its Hausdorff dimension $d_H(\Lambda_\Gamma)$ reaching the maximal value $d$. In this extreme case we found an interesting physical interpretation of a mathematical relation between the volume of cusped hyperbolic manifolds and the number of cusps on them, in terms of the space-time uncertainty principle.

In order for conformal fields on $M'$ to have an intrinsic description (required for formulating the $AdS/CFT$ correspondence), we argued that the spectrum of conformal dimensions should be bounded from below by a critical exponent $\delta(\Gamma)$, which is roughly the same as the Hausdorff dimension $d_H(\Lambda_\Gamma)$ of the limit set. We have given two arguments, one which was based on the boundary theory and the other rested on bulk supergravity, both led to the same bound. This bound will have visible effect provided it exceeds another bound, $d/2$, established in spaces without taking the quotient. The critical exponent $\delta(\Gamma)$ gives a natural measure of the "size" of $\Gamma$, so when $\Gamma$ becomes sufficiently large the scaling dimension of the most relevant operators will increase. In the extreme case $\delta(\Gamma) = d$ we could have a boundary theory without relevant operators.

We also computed the scalar two-point function by reducing the action of massive scalars in $X$ to a bilinear term on the quotient boundary $M'$. The result turned out to be consistent with conformal invariance of the boundary theory. In this computation, we have restricted ourselves to the simplest case $M' = D_\Gamma/\Gamma$ and ignored the possibility that cusps of $\Gamma$ can live outside the domain of discontinuity. If there are cusps touching the limit set, then our computation would be complicated by possible contributions from the "cusp forms". In general, these contributions are expected to be not negligible since the hand-wave argument presented at the end of section 3 suggests that a neighbourhood of a cusp, even it is very small, could contain nontrivial physical information. In a sense, cusps look like black holes, both of which render the manifold incomplete. It would be interesting to know more about the role of cusps in the $AdS/CFT$ correspondence.

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**Note added.** After this paper was submitted as e-print, I became aware of an existing work [26] where a similar connection between the $AdS/CFT$ correspondence and geometry and topology of hyperbolic manifolds was observed. Closely related results were also reported in the earlier work [27], in which the quotient generalization of the $AdS/CFT$ duality was first proposed. I would like to thank G. Horowitz for pointing out the reference [27] to me.

## A. Γ-invariant truncating

In this and the following appendix, we will give further clarification of some issues addressed in the main body of this paper.

At the beginning of section 3, we mentioned that in the quotient generalization of the $AdS/CFT$ correspondence, the boundary theory associated to supergravity on $X$ should be identified with a Γ-invariant truncation (or projection) of the usual boundray CFT, dual to the original $AdS$ supergravity without taking the quotient. We now want to show that this is indeed a reasonable speculation. We will first consider Γ-invariant projection of the usual boundary CFT in some detail, and then compare it to the boundary theory explored in the text.

For the usual boundary CFT defined on $\mathbf{S}^d$, a scalar operator $\widehat{\mathcal{O}}(\mathbf{x})$ of conformal dimension $\Delta$ transforms under $f \in \mathrm{SO}(d+1,1)$ as

$$\widehat{\mathcal{O}}(\mathbf{x}) \longrightarrow \widehat{\mathcal{O}}^f(\mathbf{x}) \equiv \mathcal{U}_f^{-1}\, \widehat{\mathcal{O}}(\mathbf{x})\, \mathcal{U}_f = |f'(\mathbf{x})|^\Delta\, \widehat{\mathcal{O}}(f\mathbf{x})\,, \tag{A.1}$$

which has the familiar two-point correlation function

$$\langle \widehat{\mathcal{O}}(\mathbf{x})\widehat{\mathcal{O}}(\mathbf{y})\rangle = \frac{const.}{|\mathbf{x}-\mathbf{y}|^{2\Delta}}\,. \tag{A.2}$$

Consider now the Γ-invariant projection of $\widehat{\mathcal{O}}(\mathbf{x})$ defined by the Poincaré series:

$$\mathcal{O}(\mathbf{x}) = \sum_{\gamma\in\Gamma} |\gamma'(\mathbf{x})|^\Delta\, \widehat{\mathcal{O}}(\gamma\mathbf{x})\,. \tag{A.3}$$

Under the action of a conformal transformation $f \in \mathrm{SO}(d+1,1)$, the change in the projected field (A.3) will be computed according to (A.1):

$$\begin{aligned}
\mathcal{O}(\mathbf{x}) \;\to\; \mathcal{O}^f(\mathbf{x}) &\equiv \mathcal{U}_f^{-1}\, \mathcal{O}(\mathbf{x})\, \mathcal{U}_f = \sum_{\gamma\in\Gamma} |\gamma'(\mathbf{x})|^\Delta\, \mathcal{U}_f^{-1}\, \widehat{\mathcal{O}}(\gamma\mathbf{x})\, \mathcal{U}_f \\
&= \sum_{\gamma\in\Gamma} |f'(\gamma\mathbf{x})|^\Delta \cdot |\gamma'(\mathbf{x})|^\Delta\, \widehat{\mathcal{O}}(f\gamma\mathbf{x}) \\
&= \sum_{\gamma\in\Gamma} |(f\gamma)'(\mathbf{x})|^\Delta\, \widehat{\mathcal{O}}(f\gamma\mathbf{x}) \\
&= |f'(\mathbf{x})|^\Delta \sum_{\gamma\in\Gamma} |(f\gamma f^{-1})'(f\mathbf{x})|^\Delta\, \widehat{\mathcal{O}}(f\gamma f^{-1}\cdot f\mathbf{x}) \\
&= |f'(\mathbf{x})|^\Delta \sum_{\gamma\in f\Gamma f^{-1}} |\gamma'(f\mathbf{x})|^\Delta\, \widehat{\mathcal{O}}(\gamma f\mathbf{x})\,.
\end{aligned} \tag{A.4}$$

The above computation tells us two things: First, if $f \in \Gamma$, then $f\gamma \in \Gamma$ and we can rearrange the sum in the third line of (A.4) to derive $\mathcal{O}^f(\mathbf{x}) = \mathcal{O}(\mathbf{x})$, which clearly indicates that the projection (A.3) indeed defines a $\Gamma$-invariant field. Second, given $f \in \mathrm{SO}(d+1,1)$, the last identity in (A.4) shows that the projected operator $\mathcal{O}(\mathbf{x})$ will transform as a conformal field of dimension $\Delta$

$$\mathcal{O}(\mathbf{x}) \to \mathcal{O}^f(\mathbf{x}) = |f'(\mathbf{x})|^\Delta \mathcal{O}(f\mathbf{x}) = |f'(\mathbf{x})|^\Delta \sum_{\gamma\in\Gamma} |\gamma'(f\mathbf{x})|^\Delta \, \widehat{\mathcal{O}}(\gamma f\mathbf{x}) \tag{A.5}$$

if and only if $f\Gamma f^{-1} = \Gamma$. This implies that performing the projection (A.3) will break the full conformal group $\mathrm{SO}(d+1,1)$ to a subgroup defined by (2.4). Thus, on symmetry grounds, the $\Gamma$-invariant truncation of the usual boundary CFT should agree with the boundary theory dual to supergravity on $X$, as both theories have the same conformal group.

In addition to this, the scalar two-point function of the truncated theory can be evaluated by substituting (A.3) into (A.2):

$$\langle \mathcal{O}(\mathbf{x})\mathcal{O}(\mathbf{y})\rangle \propto \sum_{\gamma_1,\gamma_2\in\Gamma} \frac{|\gamma_1'(\mathbf{x})|^\Delta |\gamma_2'(\mathbf{y})|^\Delta}{|\gamma_1\mathbf{x} - \gamma_2\mathbf{y}|^{2\Delta}} = \sum_{\gamma_1,\gamma_2\in\Gamma} \frac{|(\gamma_1^{-1}\gamma_2)'(\mathbf{y})|^\Delta}{|\mathbf{x} - \gamma_1^{-1}\gamma_2\mathbf{y}|^{2\Delta}}\,. \tag{A.6}$$

Thus $< \mathcal{O}(\mathbf{x})\mathcal{O}(\mathbf{y}) >$ coincides with the two-point function $G(\mathbf{x},\mathbf{y})$ derived in (5.11), up to an overall infinite factor $|\Gamma|$ that can be absorbed by normalization of the vacuum state in the truncated theory. One expects that such a coincidence also holds for higher point functions.

The foregoing discussion suggests that the boundary theory studied in the text is not very different from the boundary CFT dual to the usual $AdS$ supergravity; it is likely that the former can be constructed by $\Gamma$-invariant truncating of the latter. The only subtleties in this construction may arise as whether the projection (A.3) or its "topological dual" is a mathematically well-defined object. Since $\mathcal{U}_\gamma \mathcal{O}(\mathbf{x})\mathcal{U}_\gamma^{-1} = \mathcal{O}(\mathbf{x})$ for any $\gamma \in \Gamma$, naively the operator (A.3) would be defined on the orbit space $M = \mathbf{S}^d/\Gamma$; but as we saw in the main body of this paper, $M$ in general is not a Hausdorff space and in order to formulate a reasonable boundary CFT, we have to restrict ourselves to the subspace $M' \subset M$. Moreover, in the usual $AdS/CFT$ correspondence the scalar operator $\widehat{\mathcal{O}}(\mathbf{x})$ on $\mathbf{S}^d$ is associated to some conformal density $d\widehat{\mu}$ of dimension $\Delta$ through the conformally invariant coupling $\int \widehat{\mathcal{O}}(\mathbf{x})d\widehat{\mu}(\mathbf{x})$. When one extends this coupling to the truncated operator (A.3), one has to consider the integral

$$\int \mathcal{O}(\mathbf{x})d\widehat{\mu}(\mathbf{x}) = \int \sum_{\gamma\in\Gamma} |\gamma'(\mathbf{x})|^\Delta \widehat{\mathcal{O}}(\gamma\mathbf{x})d\widehat{\mu}(\mathbf{x}) = \int \widehat{\mathcal{O}}(\mathbf{y}) \sum_{\gamma\in\Gamma} |(\gamma^{-1})'(\mathbf{y})|^{-\Delta} d\widehat{\mu}(\gamma^{-1}\mathbf{y})\,, \tag{A.7}$$

where we have used the identity $|\gamma'(\gamma^{-1}\mathbf{y})| \cdot |(\gamma^{-1})'(\mathbf{y})| = 1$. This forces us to consider the $\Gamma$-invariant projection $d\mu(\mathbf{x}) = \sum_{\gamma\in\Gamma} |\gamma'(\mathbf{x})|^{-\Delta} d\widehat{\mu}(\gamma\mathbf{x})$ of the conformal density, whose existence requires the condition $\Delta \geq \delta(\Gamma)$ as described in section 4. Thus, modulo these subtleties, the $\Gamma$-invariant truncation of the usual CFT dual to supergravity on $\mathbf{H}^{d+1}$ should be identified with the boundary theory associated to the bulk theory on $X$.

## B. Information density in the boundary

We turn now to an estimation of information density in the boundary CFT dual to supergravity on the quotient bulk $X$. For simplicity, let us restrict ourselves to $d = 4$ dimensions; we have just seen that the underlying boundary theory on $M'$ can be regarded as the $\Gamma$-invariant truncation of the usual super Yang-Mills theory defined on $\mathbf{S}^4$. This observation allows us to estimate the number of degrees of freedom in a way similar to that presented in [12]. As in section 5, we shall focus mainly on the case where cusps of $\Gamma$ do not touch the limit set, so that $M' = D_\Gamma/\Gamma$. Contributions from cusps to the amount of information may be investigated along the line of establishing the space-time uncertainty relation (3.11); see also the ending paragraph of this appendix for a related discussion. The results derived below might have a generalization to other dimensions as well, by incorporating the analysis given in [13].

Locally, the hyperbolic metric on $X = \mathbf{H}^5/\Gamma$ can be written in the form

$$ds^2 = \frac{R^2}{x_0^2}\,(dx_0^2 + |d\mathbf{x}|^2)\,, \qquad R \equiv l_s(2\pi g_s N)^{1/4}\,, \qquad \mathbf{x} \equiv (T, X_1, X_2, X_3)\,, \tag{B.1}$$

where the dimensionless coordinates $(x_0, \mathbf{x})$ describe points of a certain fundamental domain $\mathcal{G} \subset \mathbf{H}^5$ for $\Gamma$. Let $\bar{\mathcal{G}}$ denote the closure of $\mathcal{G}$. The set $\bar{\mathcal{G}} \cap D_\Gamma$ then gives a fundamental domain for $\Gamma$ living in $D_\Gamma$, which can be identified with the quotient boundary $M'$. Since we are considering the nontrivial case where the boundary set has a non-zero measure, the hyperbolic volume of $X$ diverges and thus the bulk theory needs to be regulated. One simple way to do so is to take the horosphere $T_\epsilon$ with $\epsilon \ll 1$ as in (5.8), replacing the boundary set $\bar{\mathcal{G}} \cap D_\Gamma$ by $\bar{\mathcal{G}} \cap T_\epsilon$. The resulting space has the induced metric

$$ds^2 = \frac{R^2}{\epsilon^2}\,(dT^2 + dX_1^2 + dX_2^2 + dX_3^2)\,. \tag{B.2}$$

Consequently, the spacial area on the boundary set is regulated by

$$A = \frac{R^3}{\epsilon^3}\int dX_1\,dX_2\,dX_3 \approx \frac{R^3}{\epsilon^3}\,(\Delta U)^3\,, \tag{B.3}$$

where $\Delta U$ denotes the (dimensionless) Euclidean length scale of the regulated space $\bar{\mathcal{G}} \cap T_\epsilon$. The quantity $1/\epsilon$ may be thought of as an IR cutoff of the bulk since the hyperbolic volume of $X_\epsilon \equiv \{x \in X | x_0 > \epsilon\}$ becomes finite

$$\mathrm{Vol}(X_\epsilon) = R^5 \int_\epsilon^\infty \frac{dx_0}{x_0^5} \int_{\bar{\mathcal{G}} \cap T_\epsilon} dT\, dX_1\, dX_2\, dX_3 \;\approx\; R^5 \left(\frac{\Delta U}{\epsilon}\right)^4 , \tag{B.4}$$

and the boundary of $X_\epsilon$ can be identified with $\bar{\mathcal{G}} \cap T_\epsilon$.

We are now ready to estimate the information density. To test holography in the quotient generalization of the $AdS/CFT$ correspondence, one may assume that regulating the spacial area on the boundary set amounts to regulating the super Yang-Mills theory on $M' \sim \bar{\mathcal{G}} \cap D_\Gamma$ with an UV cutoff $\epsilon$. This assumption can be verified directly by comparing the roles of the regulator $\epsilon$ in both the boundary- and bulk- theories, as described in section 2 of ref. [12]. For an independent check, one may also determine whether such an assumption will lead to the correct information density, which can be done following the discussion in section 3 of [12]. Thus, let us think of the regulated boundary theory as a theory living on a lattice whose cells have the size $\epsilon$. The total number of cells in the spacial directions can be estimated roughly by $(\Delta U)^3/\epsilon^3$, so for a large $N$ gauge theory on the lattice, the number of degrees of freedom $N_{dof}$ behaves as

$$N_{dof} \sim \frac{N^2 (\Delta U)^3}{\epsilon^3} . \tag{B.5}$$

This together with (B.3) as well as the relation between $R$ and $N$ given in (B.1) leads to the desired result [12]

$$N_{dof} \sim \frac{A}{G_5} , \tag{B.6}$$

where $G_5 \sim l_s^8 g_s^2 R^{-5}$ is the 5-dimensional Newton's constant. Since each degree of freedom stores a single bit of information, the information density is given by $N_{dof}/A \sim 1/G_5$, and therefore it cannot exceed one bit per Planck area. Note that as $\epsilon \to 0$, both the area (B.3) and the number of degrees of freedom (B.5) will blow up. However, even when such a cutoff is removed, the information density remains finite, bounded by the inverse Planck area $1/G_5$.

Let us end with a brief discussion on the extreme case mentioned in section 3, where each component of $D_\Gamma = \mathbf{S}^4 - \Lambda_\Gamma$ collapses. Suppose that we have $p$ components of the boundary set $\bar{\mathcal{G}} \cap D_\Gamma$ and when $D_\Gamma$ collapses, this boundary set becomes $p$ cusps living in $\Lambda_\Gamma$. In that case, the length scale $\Delta U$ in (B.3)–(B.5) no longer keeps finite when the cutoff is removed, since as $\epsilon \to 0$, the Euclidean volume of the regulated boundary $\bar{\mathcal{G}} \cap T_\epsilon$

$$\int_{\bar{\mathcal{G}} \cap T_\epsilon} dT\, dX_1\, dX_2\, dX_3 \approx (\Delta U)^4 , \tag{B.7}$$

will be contributed from $p$ connected parts, each of which behaves as $\epsilon^4$ and thus $\Delta U \sim p^{1/4} \epsilon \to 0$. It follows from this and (B.3)–(B.5) that

$$A \sim p^{3/4} R^3 , \qquad \mathrm{Vol}(X_\epsilon) \sim p R^5 , \qquad N_{dof} \sim p^{3/4} N^2, \tag{B.8}$$

so that removing the cutoff $\epsilon$ will not render the regulated quantities $A$, $\mathrm{Vol}(X_\epsilon)$ as well as $N_{dof}$ divergent. This confirms our observation made in section 3 that the extreme case itself can be viewed as a regulated theory, in which no further regulating procedure (e.g. introducing a cutoff $\epsilon$ as above) is needed. One may compare further $A \sim p^{3/4} R^3$ in (B.8) with the usual expression for the regulated area [12] $A \sim R^3/\delta^3$, where $\delta$ is the dimensionless size of each cell on the discretized boundary. The comparison gives $\delta \sim p^{-1/4}$, which agrees with our earlier estimation (3.8). The volume of $X$ estimated in (B.8) is also consistent with the mathematical relation given in (3.7). The amount of information grows like $N_{dof} \propto p^{3/4}$ when the number of cusps $p$ becomes large, showing that the cusps are indeed capable of storing information. Finally, the factor $p^{3/4}$ in (B.8) will be cancelled when computing the information density, which again leads to the expected result $N_{dof}/A \sim 1/G_5$. The considerations here may have a possible extension to the hybrid case in which only some of the components of $D_\Gamma$ collapse, and others do not.

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